McDuff-Siegel Capacities

Shah Faisal
Berlin Mathematical School

March 26, 2023

These are notes from my talk in the symplectic geometry seminar in the working group of Klaus Mohnke, Chris Wendl, and Thomas Walpuski in Berlin. In this talk, I explain some interesting properties of the symplectic capacities that appeared in the paper by McDuff and Siegel, Symplectic capacities, unperturbed curves, and convex toric domains, [4].

1 Set-up

- (M, ω) denotes a closed symplectic manifold of dimension $2n$ which is semi-positive, i.e

$$\forall A \in \pi_2(M) \text{ with } \omega(A) > 0 \text{ and } c_1(A) \geq 3 - n \implies c_1(A) \geq 0.$$

- For $p \in M$, D_p denotes a local co-dimension 2 symplectic sub-manifold

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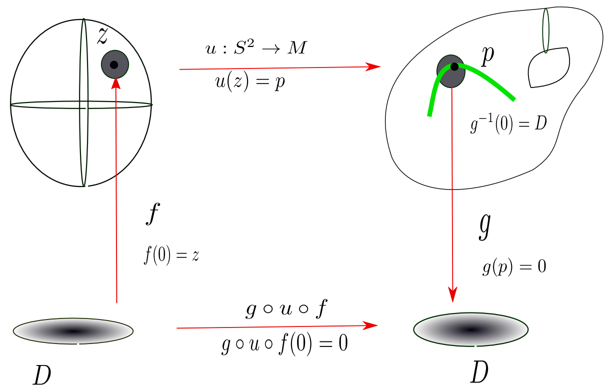
$$\mathcal{J}(M, D_p) := \left\{ J : \begin{cases} J \text{ is } \omega\text{-compatible almost complex structure on } M \\ J \text{ is integrable near } p \\ D_p \text{ is } J\text{-holomorphic} \end{cases} \right\}$$

2 Rational curves with local tangency constraints

Let $J \in \mathcal{J}(M, D_p)$ and $u : S^2 \to (M, J)$ be a J-holomorphic curve with $u(z) = p$ for some $z \in S^2$. For holomorphic chart f and holomorphic function g describing D_p consider the following diagram.



Definition 2.1. Let $k \in \mathbb{Z}_{\geq 1}$, u satisfies the tangency constraint $\ll \mathcal{T}^{k-1}p \gg$ at z w.r.t to D if

$$\frac{d^i(g \circ u \circ f)}{d^i z}|_{z=0} = 0,$$

for all $i = 0, 1, \ldots, k-1$. Ord(u, z, D):= the maximal such k. For details see Cieliebak-Mohnke [2].

Remark 2.2. For $k = 2$, $\ll \mathcal{T}^1 p \gg$ means

$$du(T_z S^2) \subset T_p D \text{ (co-dim 2 subspace of } T_p M).$$

Definition 2.3. Let $k \in \mathbb{Z}_{\geq 1}$, $A \in H_2(M, \mathbb{Z})$ and $J \in \mathcal{J}(M, D_p)$. Define

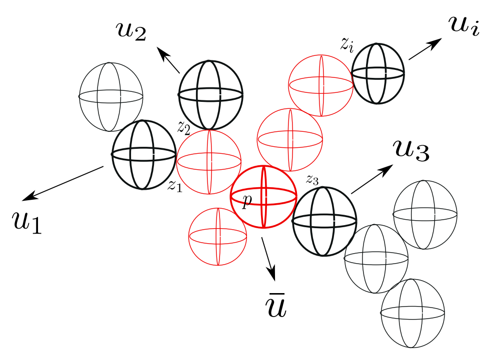
$$\mathcal{M}^J_{M,A} \ll \mathcal{T}^{k-1}p \gg := \left\{ (u, z) : \begin{cases} u : S^2 \to M \\ du \circ i = J \circ du \\ \text{u satisfies} \ll \mathcal{T}^{k-1}p \gg \text{ at } z_0 \\ u_*[S^2] = A \end{cases} \right\} / \sim$$

$(u_1, z_1) \sim (u_2, z_2)$ if and only if $(u_1, z_1) = (u_2 \circ \phi, \phi^{-1}(z_2))$ for some $\phi \in \mathrm{Aut(S^2)}$. $\widehat{\mathcal{M}}^J_{M,A} \ll \mathcal{T}^{k-1}p \gg$ denotes the parameterized moduli space.

How does the Gromov-compactness of the above moduli space look like? The following lemma of Cieliebak and Mohnke answers it.

Lemma 2.4. *(Cieliebak-Mohnke [2], special case of lemma 7.2) Let* $u_n \in \widehat{\mathcal{M}}^J_{M,A} \ll \mathcal{T}^{k-1}p \gg$ *be a sequence de-generates to a nodal configuration* u *in the Gromov topology. Suppose the constrained marked point lies on a ghost component* $\bar{u}$ *in* u*. Let* $\{u_i\}_{i=1,2,\ldots q}$ *be the non-constant components of* u *that are attached to* $\bar{u}$ *directly* *or via some ghost components. Let z_i be the special point of u_i that realize the node with $\bar{u}$ or with a ghost compment attached to $\bar{u}$. Then*

$$\sum_{i=1}^{q} \mathrm{Ord}(u_i, z_i, D_p) \geq k.$$



In the picture, the red spheres are the ghosts that shares a nod with the ghost(deep red) that inherits the constrained marked point.

Curves with local tangency constraints leads to a definition of a variant of Gromov-Witten invariants:

Theorem 2.5. *(Cieliebak-Mohnke [2], 2007, special case)*
Suppose (M, ω) is closed and semi-positive.

- *For generic $J \in \mathcal{J}(M, D_p)$, the moduli space*

$$\mathcal{M}^J_{M,A} \ll \mathcal{T}^{c_1(A)-2} p \gg$$

is a oriented compact smooth zero-dimensional manifold.

- *The signed count*

$$N_{M,A} \ll \mathcal{T}^{c_1(A)-2} p \gg := \# \mathcal{M}^J_{M,A} \ll \mathcal{T}^{c_1(A)-2} p \gg$$

does not depend on the choice of p, D_p, and J.

Theorem 2.6. *(Cieliebak-Mohnke [3], 2014)*

$$N_{\mathbb{CP}^n, [\mathbb{CP}^1]} \ll \mathcal{T}^{n-1} p \gg = (n-1)!.$$

Theorem 2.7. *(McDuff-Siegel, 2019)*

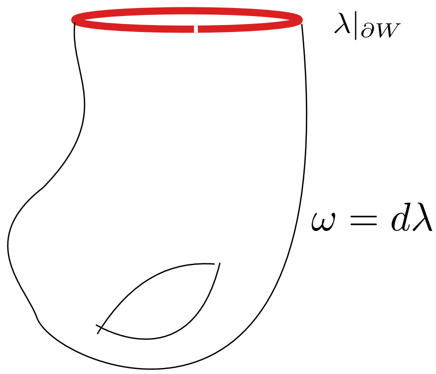
$$N_{\mathbb{CP}^2, d[\mathbb{CP}^1]} \ll \mathcal{T}^{3d-2} p \gg \neq 0$$

and can be computed using the algorithm of Göttsche–Pandharipande ([5], Theorem 3.6).

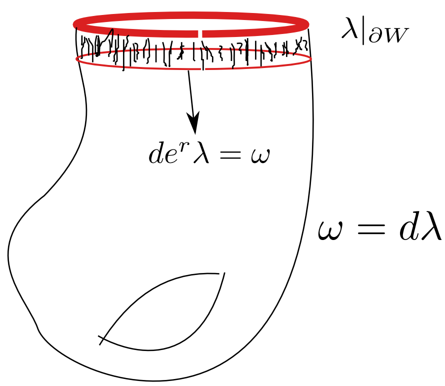
3 Liouville domains

Definition 3.1. (Liouville domain) A Liouville domain is a triple (W, ω, λ) where

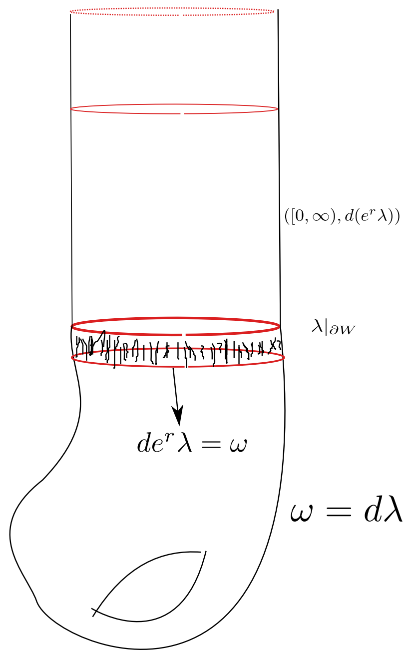
- (W, ω) is a compact symplectic manifold
- $\omega = d\lambda$
- ∂W is positive, i.e X defined by $\omega(X, .) = \lambda$ point outward along ∂W.



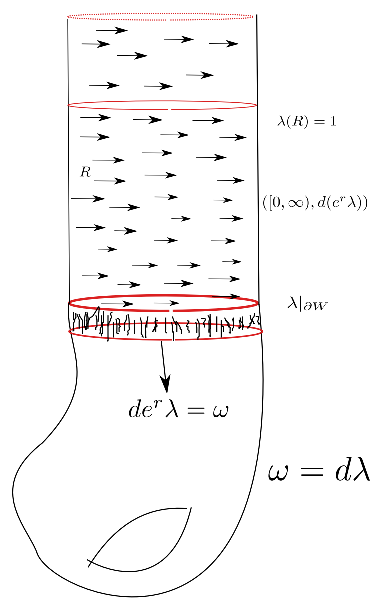
By the symplectic collar neighborhood theorem, ω looks like $d(e^r\lambda)$ on the dark region.



By attaching the half symplectic cylinder $([0, \infty), d(e^r\lambda))$ to the configuration above we get



The above configuration is called the symplectic completion of W. We will denote it by $\widehat{W}$. The horizontal vector field R dual to the λ, shown below, is called the Reeb vector field of λ.



Definition 3.2. (Admissible almost complex structures) An almost complex structure J on $\widehat{W}$ is admissible if

- it is compatible with the symplectic form
- it is r-translation invariant in a neighborhood of the cylindrical end
- it preserves $\xi := \ker(\lambda)$ and maps ∂_r to the Reeb vector field R_α

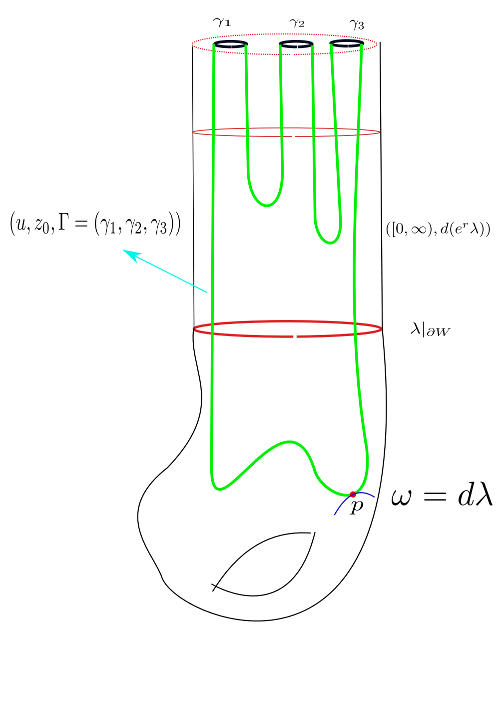
Definition 3.3. Let $p \in \mathrm{Int}(W)$, define

$$\mathcal{J}(\widehat{W}, D_p) := \left\{ J : \begin{cases} J \text{ is admissible a.c.s on } \widehat{W} \\ J \text{ is integrable near } p \\ D_p \text{ is } J\text{-holomorphic} \end{cases} \right\}$$

Definition 3.4. Let $\Gamma := (\gamma_1, \gamma_2, \dots, \gamma_l)$ be a tuple of closed Reeb orbits on ∂W. Let $k \in \mathbb{Z}_{\geq 1}$, and $J \in J(\widehat{W}, D)$. Define

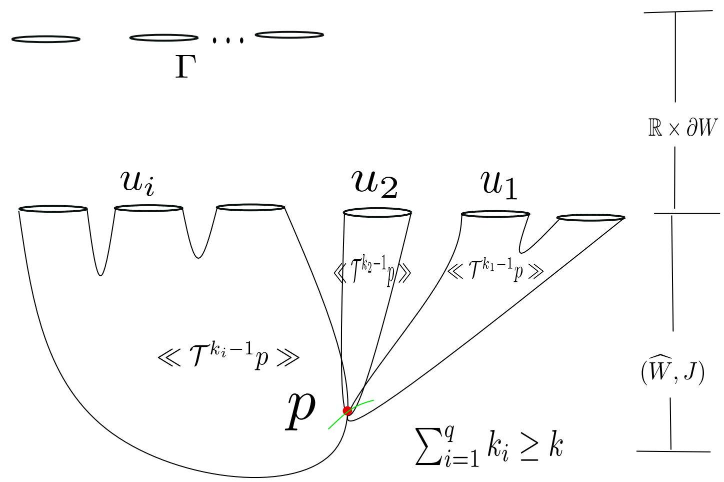
$$\mathcal{M}_W^J(\Gamma) \ll \mathcal{T}^{k-1}p \gg := \left\{ (u, z_0) : \begin{cases} u : S^2 \setminus \{z_1, \dots, z_l\} \to \widehat{W} \\ du \circ i = J \circ du \\ \text{u satisfies} \ll \mathcal{T}^{k-1}p \gg \text{at } z_0 \\ \text{u is asymptotic to } \Gamma \end{cases} \right\}$$

This moduli space contains curves that look like:



Definition 3.5. (Compactification of $\mathcal{M}_W^J(\Gamma) \ll \mathcal{T}^{k-1}p \gg$)
By the Cieliebak-Mohnke lemma above, we define

$$\overline{\overline{\mathcal{M}}}_W^J(\Gamma) \ll \mathcal{T}^{k-1}p \gg := \mathcal{M}_W^J(\Gamma) \ll \mathcal{T}^{k-1}p \gg \bigcup \{\text{buildings that look like.. shown below}\}$$



4 McDuff-Siegel Capacities

Definition 4.1. (Symplectic Capacity) Let $\mathcal{C}$ denotes a class of symplectic manifolds. A symplectic capacity is a pair $(\mathcal{C}, \alpha)$, where α is a function

$$\alpha : \mathcal{C} \to [0, \infty]$$

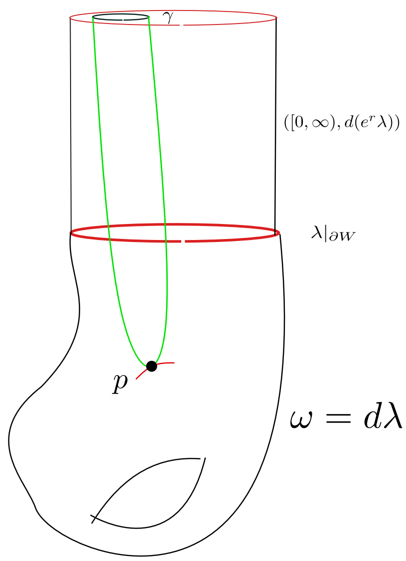
such that:

- **Scaling:** For any $a > 0$, $\alpha(M, a\omega) = a\alpha(M, \omega)$.
- **Symplectic embedding monotonicity:** If there is an quidimensional symplectic embedding (possibly with some extra conditions) $i : (M_1, \omega_1) \to (M_2, \omega_2)$, then
$$\alpha(M_1, \omega_1) \leq \alpha(M_2, \omega_2).$$
- **Non-triviality:** $0 < \alpha(\mathbb{B}^{2n}(1), \omega_0)$ and
$$0 < \alpha(\mathbb{B}^2(1) \times \mathbb{C}^{(n-1)}, \omega_0) < \infty.$$

Definition 4.2. (McDuff-Siegel Capacities [4], 2022) Let (W, λ) be a non-degenerated Liouville domain. Let D_p be a smooth local symplectic divisor passing through $p \in \operatorname{Int} X$. For $k \in \mathbb{N}$, define

$$\mathrm{MS}^1_k(W) := \sup_{J\in\mathcal{J}(\widehat{W},D_p)} \inf_{\gamma} \ \mathrm{period}(\gamma) \in [0,\infty]$$

where the infimum is taken over all periodic Reeb orbits γ for which $\mathcal{M}^J_W(\gamma) \ll \mathcal{T}^{k-1}p \gg \neq \emptyset$. Because the symplectic structure on W is exact, by stokes' theorem the infimum above is taken over the energy of asymptotic J-holomorphic disks in $\widehat{W}$:



Theorem 4.3. *(McDuff-Siegel [4], 2022) For all $k \in \mathbb{N}$, the number $\mathrm{MS}^1_k(W)$ does not depend on the choice of (p, D_p) and J, and it is a symplectomorphism invariant. Moreover,*

$$\mathrm{MS}^1_1(W) \leq \mathrm{MS}^1_2(W) \leq \mathrm{MS}^1_3(W) \leq \dots$$

Proof : Let (p, D_p) and $(p', D_{p'})$ be two choices. Choose a symplectomorphism. $\phi : W \to W$ such that $\phi(p) = p'$ and $\phi(D) = D'$. Then we have bijections

$$\phi^* : \mathcal{J}(\widehat{W}, D'_{p'}) \to \mathcal{J}(\widehat{W}, D_p) : J \to (d\phi)^{-1} \circ J \circ d\phi$$

and

$$\phi : \mathcal{M}^J_W(\gamma) \ll \mathcal{T}^{k-1}p' \gg \to \mathcal{M}^J_W(\gamma) \ll \mathcal{T}^{k-1}p \gg : u \to \phi^{-1} \circ u.$$

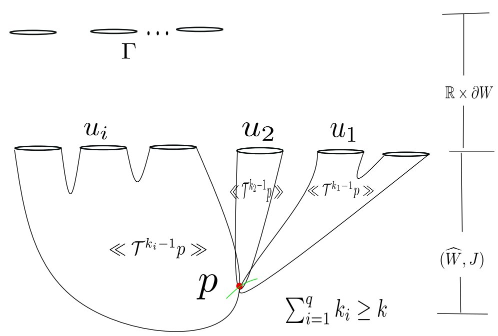
Moreover, the monotonicity w.r.t k follows from

$$\mathcal{M}^J_W(\gamma) \ll \mathcal{T}^{k-2}p \gg \subseteq \mathcal{M}^J_W(\gamma) \ll \mathcal{T}^{k-1}p \gg .$$

Definition 4.4. (McDuff-Siegel Capacities [4], 2022) Let (W, λ) be a Liouville domain. Let D_p be a smooth local symplectic divisor passing through $p \in \operatorname{Int} X$. For $m, k \in \mathbb{N}$, define

$$\mathrm{MS}_k^m(W) := \sup_{J \in \mathcal{J}(\widehat{W}, D_p)} \inf_{\Gamma} \sum_{\gamma \in \Gamma} \mathrm{period}(\gamma) \in [0, \infty]$$

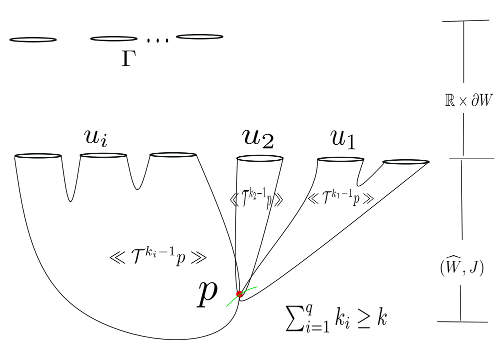
where the infimum is taken over all tuples closed Reeb orbits Γ with $\#\Gamma \leq m$ for which $\overline{\overline{\mathcal{M}}}_W^J(\Gamma) \ll \mathcal{T}^{k-1} p \gg \neq \emptyset$. The infimum is taken over the energy of holomorphic buildings that look like:



Definition 4.5. (McDuff-Siegel Capacities [4], 2022) Let (W, λ) be a Liouville domain. Let D be a smooth local symplectic divisor passing through $p \in \operatorname{Int} X$. For $k \in \mathbb{N}$, define

$$\mathrm{MS}_k(W) := \sup_{J \in \mathcal{J}(\widehat{W}, D_p)} \inf_{\Gamma} \sum_{\gamma \in \Gamma} \mathrm{period}(\gamma) \in [0, \infty]$$

where the infimum is taken over all tuples closed Reeb orbits Γ for which $\overline{\overline{\mathcal{M}}}_W^J(\Gamma) \ll \mathcal{T}^{k-1} p \gg \neq \emptyset$. The infimum is taken over the energy of holomorphic buildings that look like:



Definition 4.6. (McDuff-Siegel Capacities [4], 2022) Let (M, ω) be any symplectic manifold. For $k \in \mathbb{N}$, define

$$\mathrm{MS}_k(M, \omega) := \sup_W \mathrm{MS}_k(W) \in [0, \infty]$$

where the sup. is is taken over all Liouville domains (W, λ) which can be symplectically embedded into M.

Theorem 4.7. *(McDuff-Siegel [4], 2022) For all $k \in \mathbb{N}$, the number $\mathrm{MS}_k(M, \omega)$ does not depend on the choice of (p, D_p) and J, and it is a symplectomorphism invariant. Moreover,*

$$\mathrm{MS}_1(M, \omega) \leq \mathrm{MS}_2(M, \omega) \leq \mathrm{MS}_3(M, \omega) \leq \dots$$

Theorem 4.8. *(McDuff-Siegel Capacities [4], 2022)*

- ***Scaling:*** *For any $\alpha > 0$, $\mathrm{MS}_k(M, \alpha\omega) = \alpha\, \mathrm{MS}_k(M, \omega)$ for all $k \in \mathbb{N}$.*

- ***Subadditivity:*** $\mathrm{MS}_{k_1+k_2}(M, \omega) \leq \mathrm{MS}_{k_1}(M, \omega) + \mathrm{MS}_{k_2}(M, \omega)$.

- ***Symplectic embedding monotonicity:*** *If there is an quidimensional symplectic embedding $i : (M_1, \omega_1) \to (M_2, \omega_2)$, then $\mathrm{MS}_k(M_1, \omega_1) \leq \mathrm{MS}_k(M_2, \omega_2)$ for all $k \in \mathbb{N}$.*

- ***Closed curve upper bound:*** *(M, ω) is a closed semipositive symplectic manifold satisfying $N_{M,A} \ll \mathcal{T}^{c_1(A)-2} p \gg \neq 0$ for some $A \in H_2(M, \mathbb{Z})$, then $\mathrm{MS}_{c_1(A)-1}(M, \omega) \leq \omega(A)$.*

- ***Stabilization:*** *For certain Liouville domains W we have*

  $$\mathrm{MS}_k(W \times \mathbb{C}^m) = \mathrm{MS}_k(W)$$

  for every $k, m \in \mathbb{N}$. For example, this holds when W is a four-dimensional convex toric domain.

Remark 4.9. If there is no closed Reeb orbit on the boundary of Liouville domain W, then

$$\mathrm{MS}_k(W) = \infty$$

for all $k \in \mathbb{N}$. On the other hand, if W admits a symplectic embdedding into a closed semipositive symplectic manifold M satisfying $N_{M,A} \ll \mathcal{T}^{c_1(A)-2} p \gg \neq 0$ for some $A \in H_2(M, \mathbb{Z})$. Then

$$\mathrm{MS}_{c_1(A)-1}(W) \leq \mathrm{MS}_{c_1(A)-1}(M,\omega) \leq [\omega].A < \infty.$$

Weinstein conjecture is true for W? For example

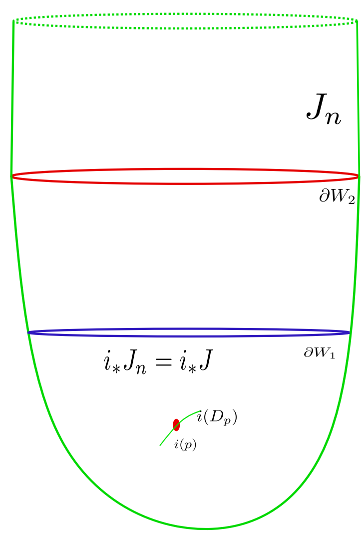
$$N_{\mathbb{CP}^n,[\mathbb{CP}^1]} \ll \mathcal{T}^{n-1}p \gg = (n-1)!.$$

So

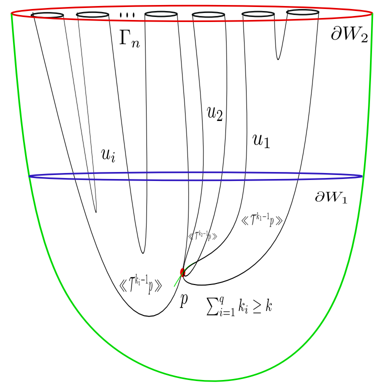
$$\mathrm{MS}_n(\mathbb{CP}^n, \omega_{FS}) \leq \pi$$

Proof. (Symplectic embedding monotonicity proof sketch)

- Suppose we have a symplectic embedding $i : (W_1, \lambda_1) \to (W_2, \lambda_2)$.
- Choose $D_p \subset \mathrm{Int}(W_1)$, then $i(D_p)$ is a local divisor in W_2.
- Given $J \in \mathcal{J}(\widehat{W}_1, D_p)$
- Let $J_n \in \mathcal{J}(\widehat{W}_2, i(D_p))$ be realizing neck-stretching along $i(\partial W_1)$ and restricts to i_*J on $i(W_1)$. See the figure below.



- Since $\mathrm{MS}_k(W_2) < \infty$, so $\overline{\overline{\mathcal{M}}}^{J_n}_{W_2}(\Gamma_n) \ll \mathcal{T}^{k-1}p \gg \neq \emptyset$. See the figure below:

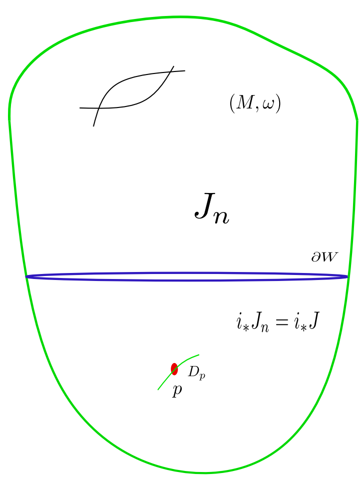


- Since the the boundary ∂W_2 is non-degenerated and $\mathrm{MS}_k(W_2) < \infty$, $\#\Gamma_n$ is bounded and becomes constant eventually.
- Let $\Gamma_n = \Gamma$, then $\sum_{\gamma \in \Gamma} \mathrm{period}(\gamma) \leq \mathrm{MS}_k(W_2)$
- $n \to \infty$ yields a configuration in $\overline{\overline{\mathcal{M}}}_{W_1}^{J_n}(\Gamma') \ll \mathcal{T}^{k-1}p \gg$ by Cieliebak-Mohnke lemma above. The energy of this building is at most $\sum_{\gamma \in \Gamma} \mathrm{period}(\gamma)$.

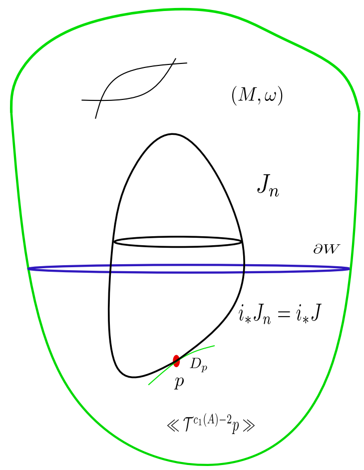
□

Proof. (Closed curve upper bound: proof sketch)

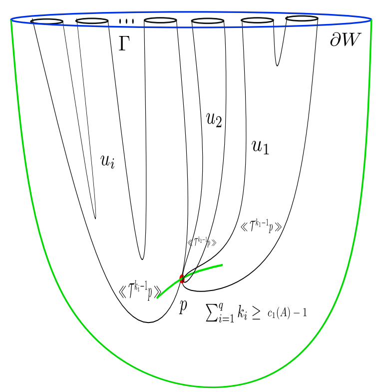
- Let $W \subseteq (M, \omega)$ be an embedded Liouville domain
- Choose $D_p \subset \mathrm{Int}(W)$
- Fix $J \in \mathcal{J}(\widehat{W}, D_p)$, let $J_n \in \mathcal{J}(M, \omega)$ be realizing neck-stretching along ∂W and restricts to J on W. See the figure below:



- Since $N_{M,A} \ll \mathcal{T}^{c_1(A)-2}p \gg \neq 0$, so $\mathcal{M}_M^{J_n} \ll \mathcal{T}^{c_1(A)-2}p \gg \neq \emptyset$.



- $n \to \infty$ yields a configuration in $\overline{\overline{\mathcal{M}}}_W^{J_n}(\Gamma) \ll \mathcal{T}^{c_(A)-2}p \gg$ by Cieliebak-Mohnke lemma above. The energy of this building is at most $\sum_{\gamma\in\Gamma} \text{period}(\gamma)$.



- Hence
$$\text{MS}_k(W) \leq \omega(A).$$

- Hence
$$\text{MS}_k(M,\omega) = \sup_{W\subseteq M} \text{MS}_k(W) \leq \omega(A).$$

$\square$

5 Applications to stabilize embedding problems

Theorem 5.1. *(McDuff-Siegel [4], 2022) Under some assumptions on Liouville domains W we have*

$$\mathrm{MS}_k(W \times \mathbb{C}^m) = \mathrm{MS}_k(W)$$

for every $m, k \in \mathbb{N}$. For example, this holds when W is a four-dimensional convex toric domain.

Question 5.2. (Stabilize embedding problem) Let W_1 and W_2 be two Liouville domains. When does there exist a symplectic embedding

$$\phi : W_1 \times \mathbb{C}^m \to W_2 \times \mathbb{C}^m?$$

If such an embedding exists, then $\mathrm{MS}_k(W_1) \leq \mathrm{MS}_k(W_2)$ for all $k \in \mathbb{N}$. This gives numerical obstructions (sometimes sharp) to the existence of such embeddings.
For an example below:

Example 5.3. (McDuff-Siegel [4], 2022) Let $1 \leq a < \infty$,

$$E(1,a) := \{(z_1, z_2) \in \mathbb{C}^2 : \pi|z_1|^2 + \pi\frac{|z_2|^2}{a} \leq 1\}$$

For $1 \geq a \leq 3/2$,

$$\mathrm{MS}_k(E(1,a)) = \begin{cases} 1 + la, \text{ for } k = 1 + 3l \text{ with } l \geq 0 \\ a + la, \text{ for } k = 2 + 3l \text{ with } l \geq 0 \\ 2 + la, \text{ for } k = 3 + 3l \text{ with } l \geq 0. \end{cases}$$

For $3/2 \leq a$,

$$\mathrm{MS}_k(E(1,a)) = \begin{cases} k, \text{ for } 1 \leq k \leq \lfloor a \rfloor \\ a + l, \text{ for } k = \lceil a \rceil + 2l \text{ with } l \geq 0 \\ \lceil a \rceil + l, \text{ for } k = \lceil a \rceil + 2l + 1 \text{ with } l \geq 0. \end{cases}$$

There exists a symplectic embedding $\phi : E(1,7) \times \mathbb{C}^m \to \mu E(1,2) \times \mathbb{C}^m$ if and only if $\mu \geq \frac{7}{4}$. This lower bound is sharp.

Proposition 5.4. *(McDuff-Siegel [4], 2022)(Stabilization lower bound)*
For any Liouville domain W, we have

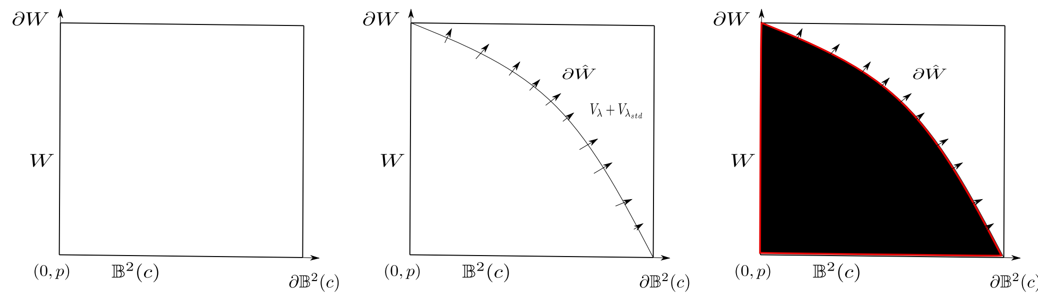
$$\mathrm{MS}_k(W) \leq \mathrm{MS}_k(W \times \mathbb{C}^m)$$

for all $k, m \geq 1$.

Remark 5.5. It is enough to prove that for each fixed k

$$\mathrm{MS}_k(W) \le \mathrm{MS}_k(W \times \mathbb{B}^2(c))$$

sufficiently large $c > 0$. The $W \times \mathbb{B}^2(c)$ is not smooth, it has singularities at $\partial W \times \partial \mathbb{D}^2$. We need to smooth it out to make a Liouville domain . The idea of McDuff and Siegel is to take a nice cut of the picture to left via a Hamiltonian function to obtain picture to the right. See [4], Lemma 3.6.2, for details.



The following theorem ensures the existence of a nice smoothing.

Theorem 5.6. *(McDuff-Siegel [4],Lemma 3.6.2, 2022) Let (W, λ) be Liouville domain. For any $\epsilon, c > 0$, there is a subdomain with smooth boundary $W \tilde{\times} \mathbb{B}^2(c) \subset W \times \mathbb{B}^2(c)$, see the figure above, such that we have*

- *the Liouville vector field $V_\lambda + V_{\lambda_{std}}$ is s outwardly transverse along $\partial(W \tilde{\times} \mathbb{B}^2(c))$*
- *$W \times \{0\} \subset W \tilde{\times} \mathbb{B}^2(c)$ and the Reeb vector field of $\partial(W \tilde{\times} \mathbb{B}^2(c))$ is tangent to $\partial W \times \{0\}$.*
- *any closed Reeb orbit of the contact form $\lambda + \lambda_{std}|_{\partial(W \tilde{\times} \mathbb{B}^2(c))}$ with period less than $c - \epsilon$ is entirely contained in $\partial W \times \{0\}$.*

Proof. (Stabilization lower bound proof sketch)

- We want to prove that for every $c > 0$ large enough

$$\mathrm{MS}_k(W) \le \mathrm{MS}_k(W \times \mathbb{B}^2(c))$$

- Since $W \tilde{\times} \mathbb{B}^2(c) \subset W \times \mathbb{B}^2(c)$, it is enough to prove for every large $c > 0$

$$\mathrm{MS}_k(W) \le \mathrm{MS}_k(W \tilde{\times} \mathbb{B}^2(c))$$

- We are done if we found a $J \in \mathcal{J}(\widehat{W \tilde{\times} \mathbb{B}^2(c)}), D_{\tilde{p}})$ such that for every tuple Γ for which
$$\overline{\overline{\mathcal{M}}}^J_{W \tilde{\times} \mathbb{B}^2(c)}(\Gamma) \ll \mathcal{T}^{k-1}\tilde{p} \gg \neq \emptyset$$
we have
$$\sum_{\gamma \in \Gamma} \mathrm{Period}(\gamma) \geq \mathrm{MS}_k(W).$$

- Choose $J_W \in \mathcal{J}(\widehat{W}, D_p)$ such that for every tuple Γ for which
$$\overline{\overline{\mathcal{M}}}^J_W(\Gamma) \ll \mathcal{T}^{k-1}p \gg \neq \emptyset$$
we have
$$\sum_{\gamma \in \Gamma} \mathrm{Period}(\gamma) \geq \mathrm{MS}_k(W).$$
Such J_W exists by definition of $\mathrm{MS}_k(W)$ and the fact that the sum of periods of finite tuples of Reeb orbits form a discrete set on the real line $\mathbb{R}$.

- Set $\tilde{p} = (p, 0) \in W \tilde{\times} \mathbb{B}^2(c)$ and the divisor $\tilde{D} := D_p \times \mathbb{B}^2(\delta)$. Choose $J \in \mathcal{J}(\widehat{W \tilde{\times} \mathbb{B}^2(c)}), D_{\tilde{p}})$ such that $J|_{\hat{W} \times \{0\}} = J_W$.

- Choose $c > \mathrm{MS}_k(W)$. For small $\epsilon > 0$ we have
$$c - \epsilon > \mathrm{MS}_k(W).$$

- Let $C \in \overline{\overline{\mathcal{M}}}^J_{\widehat{W \tilde{\times} \mathbb{B}^2(c)}}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$. If some component u of C has an end that is not asymptotic to a Reeb orbit in $\partial W \times \{0\}$, this component has energy at least c by the third bullet point in the theorem 5.6 above . Thus, the energy of C is greater than $c - \epsilon > \mathrm{MS}_k(W)$.

- If every component u of C has all ends asymptotic to Reeb orbits in $\partial W \times \{0\}$, the the whole building lies in $\widehat{W}$ and $\mathbb{R} \times \partial W$.

- So $C \in \overline{\overline{\mathcal{M}}}^J_{\widehat{W}}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$. By the choice of J in the above bullet point, the energy of C is greater than $\mathrm{MS}_k(W)$.

□

The stabilization lower bound holds for W if the value $\mathrm{MS}_k(W)$ is suported by a nice moduli space. The following makes it precise:

Theorem 5.7. *(McDuff-Siegel [4], 2022) Let W be a Liouville domain. Suppose there exist a tuple of closed Reeb orbits Γ and a relative homology class $A \in H_2(W, \Gamma, \mathbb{Z})$ such that:*

- $\sum_{\gamma\in\Gamma} \mathrm{Period}(\gamma) = \mathrm{MS}_k(W)$
- $\mathcal{M}_W^{J_W}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$ *is of index zero and regular for some* $J \in \mathcal{J}_{\mathcal{W}}(\widehat{W}, D_p)$.
- *The signed count*

$$\#\mathcal{M}_W^{J_W}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$$

  does not depend on the choice of generic $J \in \mathcal{J}(\widehat{W}, D_p)$,

then

$$\mathrm{MS}_k(W) \geq \mathrm{MS}_k(W \times \mathbb{C}^m).$$

Proof. (Rough Idea)

- It is enough to prove that for every $c > 0$

$$\mathrm{MS}_k(W) \geq \mathrm{MS}_k(W \tilde{\times} \mathbb{B}^2(c)).$$

- Choose $J_{ext} \in \mathcal{J}(\widehat{W \tilde{\times} \mathbb{B}^2(c)}), D_{\tilde{p}})$ such that $J|_{\hat{W}\times\{0\}} = J_W$.
- By construction of $\widehat{W \tilde{\times} \mathbb{B}^2(c)})$, the curves in $\mathcal{M}_W^{J_W}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$ are also J_{ext}-holomorphic, index zero and regular. The regularity survives after extending J_W requires some work to be proved. The signed count

$$\#\mathcal{M}^{J_{ext}}_{\widehat{W \tilde{\times} \mathbb{B}^2(c)})}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$$

  does not depend on the choice of regular $J \in \mathcal{J}(\widehat{W \tilde{\times} \mathbb{B}^2(c)}), D_p)$. Here one needs to appeal to the intersection theory in Moreno-Siefring [1]
- So for generic $J \in \mathcal{J}(\widehat{W \tilde{\times} \mathbb{B}^2(c)}), D_{\tilde{p}})$ there are curves in $\mathcal{M}^{J}_{\widehat{W \tilde{\times} \mathbb{B}^2(c)})}(\Gamma) \ll \mathcal{T}^{k-1}p \gg$, so

$$\mathrm{MS}_k(W) = \sum_{\gamma\in\Gamma} \mathrm{Period}(\gamma) \geq \mathrm{MS}_k(W \tilde{\times} \mathbb{B}^2(c)).$$

□

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